NASA TN D-748

TECHNICAL NOTE

D-748

ANALYSIS OF A FOUR-STATION DOPPLER TRACKING METHOD USING A SIMPLE CW BEACON

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON April 1961

NASA TN D-748

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SUMMARY

A Doppler tracking method is presented in which a very small, simple CW beacon transmitter is used with four Doppler receiving stations to obtain the position and velocity of a space research vehicle. The exact transmitter frequency need not be known, but an initial position is required, and Doppler frequencies must be measured with extreme accuracy. The errors in the system are analyzed and general formulas are derived for position and velocity errors. The proper location of receiving stations is discussed, a rule for avoiding infinite errors is given, and error charts for ideal station configurations are presented. The effect of the index of refraction is also investigated. The system is capable of determining transmitter position within 1,000 feet at a range of 200 miles.

INTRODUCTION

The accurate determination of the position and velocity of a small research vehicle is an important problem. Many elaborate methods for making this determination have been developed, most of which require either high-powered, complicated, and expensive ground equipment or complicated airborne equipment (ref. 1). There is need for a tracking system which uses extremely small simple equipment in the vehicle and moderately simple inexpensive equipment on the ground. Such a system is afforded by transmitting a low-power CW signal from a radio beacon in the vehicle and receiving Doppler signals at several ground stations.

The simplest Doppler system of this type requires a very stable transmitter and three receiving stations. The disadvantage of this method is the precision with which the transmitter frequency must be known. By using four stations, the problem becomes more involved, but it is possible to find the trajectory even though the transmitter frequency is not known precisely or even if it shifts. In this investigation the problem of finding the position by using four stations is solved. Position-error formulas are derived and optimum station

locations to minimize these errors are discussed. The accuracies required in measuring Doppler frequencies and the effect of the index of refraction are discussed. Also, the velocity and errors in velocity are found.

SYMBOLS

A,B,C Doppler receiving stations

$A_x,B_x,C_x;A_y,B_y,C_y;\ldots$ cofactors of the elements a_x, b_x, c_x; a_y, b_y, c_y; . . . respectively, in determinant E

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a,b,c distance from stations A, B, and C, respectively, to origin

$\vec{a},\vec{b},\vec{c}$ vector position of stations A, B, and C, respectively

$a_x,b_x,c_x;a_y,b_y,c_y;\ldots$ $\frac{\partial s_a}{\partial x}$, $\frac{\partial s_b}{\partial x}$, $\frac{\partial s_c}{\partial x}$; $\frac{\partial s_a}{\partial y}$, $\frac{\partial s_b}{\partial y}$, $\frac{\partial s_c}{\partial y}$; . . . (see eq. (28))

σa probable error in x_a, x_b, or x_c; y_a, y_b, or y_c; . . . when these probable errors are all equal

c_o velocity of propagation of radio waves in free space

c_s velocity of propagation at source

$$D = \begin{vmatrix} x_a & y_a & s_a \\ x_b & y_b & s_b \\ x_c & y_c & s_c \end{vmatrix}$$

$$E = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad \text{(see eq. (A2))}$$

e vapor pressure, mb

f transmitter frequency

f_a, f_b, f_c, f_o received frequencies at stations A, B, C, and O, respectively

Δf error in f

σf probable error in $(f_o - f_a)$, $(f_o - f_b)$, or $(f_o - f_c)$ when these probable errors are all equal

$d(f_o - f_a), d(f_o - f_b), d(f_o - f_c)$ differential of (or error in) $(f_o - f_a)$, $(f_o - f_b)$, and $(f_o - f_c)$

K_a, K_b, K_c quantities defined in eqs. (16), (17), and (18), respectively

l, m length of sides in rectangular configuration

N electron concentration, per cm^3

n index of refraction at transmitter

O station O at the origin

p atmospheric pressure, mb

$r \equiv r_o$ distance from origin to transmitter or distance from station O to transmitter

$\vec{r}$ vector position of transmitter relative to origin

$d\vec{r}$ differential of vector position of transmitter

σr probable error in position of transmitter

r_a, r_b, r_c distance from stations A, B, and C, respectively, to transmitter (see eqs. (10), (11), and (12))

$\vec{r}_a, \vec{r}_b, \vec{r}_c$ vector position of transmitter relative to stations A, B, and C, respectively

$r_{ao}, r_{bo}, r_{co}, r_{oo}$ initial distances from stations A, B, C, and O, respectively, to transmitter at time t_o

Δr_a error in r_a

S location of source (transmitter)

S_a, S_b, S_c — cofactors of s_a, s_b, and s_c, respectively, in determinant D

s_a, s_b, s_c — range differences, $r_a - r$ (eq. (23)), $r_b - r$, and $r_c - r$, respectively

ds_a, ds_b, ds_c — differentials of (or errors in) s_a, s_b, and s_c, respectively

σs — probable error in s_a, s_b, or s_c when these probable errors are all equal

$\sigma s_a, \sigma s_b, \sigma s_c$ — probable error in s_a, s_b, and s_c, respectively

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T — temperature, $^{\circ}K$

t — time

t_o — initial time

v_a, v_b, v_c — radial components of velocity of transmitter (source) relative to stations A, B, and C, respectively

v_x, v_y, v_z — x, y, and z components, respectively, of transmitter velocity

σv — probable error in transmitter velocity

$\sigma v_x, \sigma v_y, \sigma v_z$ — probable errors in v_x, v_y, and v_z, respectively

$X_a, Y_a; X_b, Y_b; X_c, Y_c$ — cofactors of x_a, y_a; x_b, y_b; x_c, y_c in determinant D

x, y, z — coordinates of transmitter

dx, dy, dz — differentials of (or error in) x, y, and z, respectively

$\sigma x, \sigma y, \sigma z$ — probable error in x, y, and z, respectively

$x_a, y_a, z_a; x_b, y_b, z_b \ldots$ — coordinates of stations A, B, . . .

$dx_a, dy_a, dz_a; dx_b, dy_b, dz_b \ldots$ — differentials of (or errors in) x_a, y_a, z_a; x_b, y_b, z_b; . . .

$\sigma x_a, \sigma y_a, \sigma z_a;$ $\sigma x_b, \sigma y_b, \sigma z_b;$ . . . probable errors in x_a, y_a, z_a; x_b, y_b, z_b; . . .

δ propagation error factor (see eq. (43))

ϵ error factor, ratio of probable error in $\vec{r}$ to probable error in range differences, $\sigma r/\sigma s$

η per unit error in index of refraction

λ wavelength of transmitter signal at source

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PRELIMINARY ANALYSIS

A brief analysis of the three-station method is presented. This analysis shows that a range measurement depends on the transmitter frequency. This frequency dependence can be eliminated by measuring range differences between stations. Since at least three range differences or four stations are required, an analysis of the four-station method is presented.

Three-Station Method

A nonrelativistic approximation of the Doppler equation for the frequency observed by a receiver at a fixed point A is given by

$$f_a = f\left(1 - \frac{v_a}{c_s}\right) \tag{1}$$

where v_a is the radial velocity component of the source relative to the point A, and c_s is the velocity of propagation at the source. Thus,

$$v_a = \frac{c_s}{f}\left(f - f_a\right) \tag{2}$$

or

$$v_a = \frac{c_o}{nf}\left(f - f_a\right) \tag{3}$$

$$v_a = \lambda\left(f - f_a\right) \tag{4}$$

If the initial position of the source is known, the subsequent radial distance (range) from point A can be determined from

$$r_a = r_{ao} + \int_{t_o}^{t} \lambda(f - f_a)dt \tag{5}$$

With two receiving stations A and B, the ranges r_a and r_b are then known. Figure 1 illustrates that the intersection of two spheres with radii r_a and r_b locate the transmitter somewhere along a circle; hence, an additional station C (located off the axis A, B) is required to locate the transmitter. (The intersection of the sphere C with the circle actually gives two positions, but in a practical case one of these may be disregarded since it is beneath the surface of the earth.)

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In practice the received frequencies can be measured very accurately, but the transmitter frequency f cannot be known exactly because it may shift during flight. It is difficult to keep the frequency of a very small transmitter constant under the extreme environment encountered in a rocket-propelled vehicle. From equation (5) it can be seen that an error Δf in f gives an error in r_a of

$$\Delta r_a \approx \int_{t_o}^{t} \lambda \, \Delta f \, dt \tag{6}$$

As an illustration of the magnitudes involved, consider a typical case where $f = 200$ megacycles $(\lambda \approx 5$ feet) with a constant error of 100 cycles per second. Over a time interval of 300 seconds the error in r_a would be 150,000 feet.

Four-Station Method

The error due to transmitter-frequency change may be eliminated by measuring the frequency difference between two Doppler receiving stations. Let O represent an additional receiving station located at the origin; the range difference is

$$\begin{aligned} s_a &= r_a - r_o \\ &= r_{ao} - r_{oo} + \int_{t_o}^{t} \lambda(f_o - f_a)dt \end{aligned} \tag{7}$$

There will be a small error in λ since f is not known; but f differs from f_o only by the amount of the Doppler shift which is less than 5 kilocycles for a transmitter speed of 25,000 feet/sec. Thus, if λ is based on f_o the error in s_a is less than 0.0025 percent (or less than 26 feet at a range difference of 200 miles).

There may also be a small error in s_a because of the difference in time it takes for the signal to reach different receivers. If the receiving stations are 200 miles apart, the maximum time difference is only 1 millisecond. If the velocity of the beacon suddenly changed by 25,000 ft/sec, which gives a 5-kilocycle change in Doppler signal, the maximum error in s_a would be only 25 feet. (Both these errors can be eliminated, if desired, once the position and velocity of the transmitter are determined.) Thus, by measuring the difference in frequency between two stations, the range differences can be determined without knowing the transmitter frequency.

If it is assumed that the range differences s_a, s_b, and s_c can be measured accurately, it can be seen that four receiving stations are required to locate the transmitter. In figure 2 receiving stations are shown at A, B, C, and O, with the source at S. A sphere with radius r_o is illustrated with S as the center. Since the distance from S to A is r_a, then

$$\begin{aligned} r_a &= (r_a - r_o) + r_o \\ &= s_a + r_o \end{aligned} \tag{8}$$

Hence, the sphere with radius s_a and center A is shown to be tangent to the sphere with center S. Similarly, spheres with radii s_b and s_c are shown around centers B and C. Now, given the range differences s_a, s_b, and s_c, S is located at the center of the sphere which is tangent to the spheres A, B, and C and goes through the point O. A geometric solution is thus obtained.

FOUR-STATION SOLUTION

The geometrical solution just presented is of little value in obtaining the actual solution. This solution may be obtained by writing the equations for the four ranges (fig. 3) as

$$r^2 = x^2 + y^2 + z^2 \tag{9}$$

$$r_a^2 = (x - x_a)^2 + (y - y_a)^2 + (z - z_a)^2 \qquad (10)$$

$$r_b^2 = (x - x_b)^2 + (y - y_b)^2 + (z - z_b)^2 \qquad (11)$$

$$r_c^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 \qquad (12)$$

Subtracting equation (9) from equation (10) yields

$$r_a^2 - r^2 = -2x_a x - 2y_a y - 2z_a z + x_a^2 + y_a^2 + z_a^2 \qquad (13)$$

With the substitutions

$$a^2 = x_a^2 + y_a^2 + z_a^2 \qquad (14)$$

$$\begin{aligned} r_a^2 - r^2 &= (r_a + r)(r_a - r) \\ &= (2r + s_a)s_a \\ &= s_a^2 + 2rs_a \qquad (15) \end{aligned}$$

equation (13) becomes

$$x_a x + y_a y + s_a r = \frac{a^2 - s_a^2 - 2z_a z}{2} = K_a \qquad (16)$$

Similarly,

$$x_b x + y_b y + s_b r = \frac{b^2 - s_b^2 - 2z_b z}{2} = K_b \qquad (17)$$

$$x_c x + y_c y + s_c r = \frac{c^2 - s_c^2 - 2z_c z}{2} = K_c \qquad (18)$$

In equations (16) to (18) a, x_a, y_a, z_a; b, x_b, . . . are constants which depend on station locations; s_a, s_b, and s_c are known data from the Doppler stations; x, y, and r are unknowns to be solved. Actually z is also an unknown, but since z_a, z_b, and z_c are usually small, an approximate value may be assumed in order to obtain a first approximation of x, y, and r.

The solution is then (using cofactors of determinant D)

$$x = \frac{K_a X_a + K_b X_b + K_c X_c}{D} \tag{19}$$

$$y = \frac{K_a Y_a + K_b Y_b + K_c Y_c}{D} \tag{20}$$

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$$r = \frac{K_a S_a + K_b S_b + K_c S_c}{D} \tag{21}$$

$$z = \sqrt{r^2 - x^2 - y^2} \tag{22}$$

The computed value of z may be compared with the assumed value, and if it is different it may be inserted in equations (16), (17), and (18) to find more exact values of x, y, and z. The process converges rapidly. In several actual cases where the first assumed value of z was far from the actual value, the process converged in fewer than six iterations. In a typical case in which z was actually 500,000 feet but was assumed to be zero, the errors in z for successive iterations were 32,200, 2,100, 132, and 8 feet.

ERRORS IN POSITION

This four-station solution does not always give the transmitter position. For example, consider four stations located at the corners of a square, with the transmitter along a perpendicular through the center of the square. No matter where the transmitter is located along this perpendicular, $r = r_a = r_b = r_c$ and, hence, $s_a = s_b = s_c = 0$; thus, the position is indeterminate and large errors occur in the vicinity of the perpendicular. The station configuration has a pronounced effect on the errors.

In order to investigate the errors it will be convenient to derive an error factor ϵ defined as the ratio of the probable error in the position determination to the probable error in the range-difference measurements. The error factor will be examined for conditions which cause infinite error, and error-factor contour plots of transmitter position will be given for various station configurations. The position error depends on range-difference measurements (which in turn depend on frequency measurements and propagation effects) and on measurements of station positions. All these errors will be discussed herein.

Errors Due to Range-Difference Measurements

Derivation of error factor.- Of primary importance is the error in position due to errors in the measurement of the three range differences s_a, s_b, and s_c. The quantity

$$
\begin{aligned}
s_a &= r_a - r \\
&= \sqrt{(x - x_a)^2 + (y - y_a)^2 + (z - z_a)^2} - \sqrt{x^2 + y^2 + z^2} \qquad (23)
\end{aligned}
$$

depends on the transmitter coordinates and the station locations.

Consider the variation ds_a in s_a caused by the variations dx, dy, and dz. Thus,

$$ds_a = \frac{\partial s_a}{\partial x} dx + \frac{\partial s_a}{\partial y} dy + \frac{\partial s_a}{\partial z} dz \qquad (24)$$

For simplicity this equation may be written in the form

$$ds_a = a_x \, dx + a_y \, dy + a_z \, dz \qquad (25)$$

Similarly,

$$ds_b = b_x \, dx + b_y \, dy + b_z \, dz \qquad (26)$$

and

$$ds_c = c_x \, dx + c_y \, dy + c_z \, dz \qquad (27)$$

where

$$
\begin{aligned}
a_x &= \frac{\partial s_a}{\partial x} \\
&= \frac{x - x_a}{r_a} - \frac{x}{r} \\
&= \frac{x}{r_a}\left(1 - \frac{x_a}{x} - \frac{r_a}{r}\right) \qquad (28)
\end{aligned}
$$

With the other coefficients similar in form.

Then, the deviations in x, y, and z due to ds_a, ds_b, and ds_c are given by solving equations (25), (26), and (27) as follows:

$$dx = \frac{A_x}{E} ds_a + \frac{B_x}{E} ds_b + \frac{C_x}{E} ds_c \tag{29}$$

$$dy = \frac{A_y}{E} ds_a + \frac{B_y}{E} ds_b + \frac{C_y}{E} ds_c \tag{30}$$

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$$dz = \frac{A_z}{E} ds_a + \frac{B_z}{E} ds_b + \frac{C_z}{E} ds_c \tag{31}$$

where A_x, B_y, and so forth are the cofactors of a_x, b_y, and so forth in the determinant E.

The probable errors in the range differences may be expected to be equal but uncorrelated because each station is similar but independent. Under this assumption let σs be the probable error in any one of the range differences; thus,

$$\sigma s = \sigma s_a = \sigma s_b = \sigma s_c \tag{32}$$

The probable errors in the position coordinates are then given by

$$(\sigma x)^2 = \left(A_x{}^2 + B_x{}^2 + C_x{}^2\right) \frac{(\sigma s)^2}{E^2} \tag{33}$$

$$(\sigma y)^2 = \left(A_y{}^2 + B_y{}^2 + C_y{}^2\right) \frac{(\sigma s)^2}{E^2} \tag{34}$$

$$(\sigma z)^2 = \left(A_z{}^2 + B_z{}^2 + C_z{}^2\right) \frac{(\sigma s)^2}{E^2} \tag{35}$$

Now, since

$$\left|d\vec{r}\right|^2 = dx^2 + dy^2 + dz^2 \tag{36}$$

the probable error in the vector $\vec{r}$ is given by

$$\sigma r = \sqrt{(\sigma x)^2 + (\sigma y)^2 + (\sigma z)^2} \tag{37}$$

Thus,

$$\epsilon = \frac{\sigma r}{\sigma s}$$
$$= \frac{1}{E}\sqrt{A_x^2 + B_x^2 + C_x^2 + A_y^2 + B_y^2 + C_y^2 + A_z^2 + B_z^2 + C_z^2} \qquad (38)$$

This error factor depends only on the station configuration and the transmitter position.

Infinite errors - triangle rule.- Infinite errors occur whenever E, the demoninator of the error factor, is zero. For the special case when all the receivers are in the x-y plane (this is the only case which will be considered) appendix A shows that infinite error occurs whenever the transmitter is in such a position that either $z = 0$ (i.e., in the x-y plane) or $D = 0$.

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In appendix B and figure 4 the station configurations which allow $D = 0$ are studied; from this study the following triangle rule of infinite error is established: Infinite error cannot occur (except in the x-y plane) if any one station lies within or on the triangle formed by lines joining the other three stations (fig. 5); if no station lies within such a triangle, then infinite error exists.

As an illustration, the special case of a rectangular configuration (fig. 6) is considered in appendix C, and it is shown that infinite error occurs on the two vertical planes which are perpendicular bisectors of the sides of the rectangle. For other more intricate configurations, the determination of the locations in which infinite error occurs becomes much more complex.

Error factors for various configurations.- In addition to knowing the existence of infinite errors, it is important to know the actual magnitude of the errors. It may be surmised from the infinite-error analysis that a desirable station configuration would occur when one station is at the center of an equilateral triangle formed by the other three. This equilateral-triangle configuration was analyzed by computing the error factor from equation (38). (The receiving stations have been assumed to be located on the earth's surface rather than on a flat plane.) The results are shown in figure 7 in the form of contour plots of constant error factor at various altitudes. This configuration is an ideal one for general applications where the azimuth angle may not be known in advance.

Another important configuration is the isosceles triangle with one station at the midpoint of the base. Figure 8 shows contour plots for this case (the earth's curvature is again taken into account). It may be

noticed that, in general, the overall error factor has not been improved. However, in one direction (over the apex of the triangle) the error factor has been improved at the expense of making the error factor in other directions greater. This configuration is useful when the approximate azimuth direction is known in advance. A check on the error-factor formula (eq. (38)) was made by substituting (in eqs. (19) to (22)) values for s_a, s_b, and s_c which were in error by 100 feet. This check was made for the isosceles-triangle configuration at an altitude of 500,000 feet. The results checked within 0.2 percent in the range covered along the axis 20° counterclockwise from station A (fig. 8(c)).

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One particular configuration for use at NASA Wallops Station required land stations which were readily accessible. The stations were placed at Wallops Island, Va.; Cape Hatteras, N.C.; Langley Field, Va.; and Dover, Del. Error contour plots for this case are shown in figure 9. One azimuth direction, as expected, gives infinite error, but in the direction of the predicted transmitter position (145° clockwise from north), the error factor is acceptable.

<u>Frequency accuracy</u>.- An examination of the contour plots indicates that an error factor of about 10 can be achieved. If position data within 1,000 feet are required, σs must be within 100 feet. This accuracy requires the probable error in the total number of cycles counted to be within 20 cycles ($\lambda = 5$). If this count were over a period of 300 seconds, the frequency accuracy required would 0.07 cycle per second, an accuracy of 1 part in 3×10^{9}.

It is feasible to use local oscillators which have stabilities of 1 part in 10^{9} drift per day at each station. If the oscillators are checked against each other within 8 hours, the proper accuracy could be obtained. A desirable method of calibration is to check all stations simultaneously against a common source (whose frequency need not be known exactly).

<u>Propagation errors</u>.- The index of refraction may vary and cause errors. In the troposphere the index changes because of air density variations according to (ref. 2)

$$n = 1 + \frac{77.6}{T}\left(p + \frac{4810e}{T}\right)10^{-6} \qquad (39)$$

From the ground up to 50,000 feet, the index varies from about 1.0003 to 1.0000. The index may be calculated from radiosonde data of pressure, temperature, and humidity as a function of altitude.

In the ionosphere the index varies because of ionization and depends on the electron density and frequency (ref. 3) as follows (where f is measured in kilocycles):

$$n = \sqrt{1 - \frac{81N}{f^2}} \tag{40}$$

The electron density becomes important at an altitude of about 200,000 feet; at 1×10^6 feet it attains a maximum density of the order of 1.5×10^6 per cubic centimeter (ref. 4). Thus, at 200 megacycles the index varies from 1.0000 to 0.9985 between 200,000 and 1×10^6 feet. It appears that a frequency of at least 400 megacycles should be used, in which case the variation would be only 0.0004. The index may be calculated by using electron density measurements from rocket soundings.

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The effect of the index of refraction on position determination can be seen from equation (7) where $\lambda = c_o/nf$. However, the error does not enter in the same manner as the frequency error in $(f_o - f_a)$ because the errors in s_a, s_b, and s_c due to errors in n are correlated. If the simplifying assumption is made that the per unit error in n is a constant η, and if the initial range difference is neglected, then the per unit error in s_a, s_b, and s_c is η and the error in x is (from eq. (29)) as follows:

$$dx = \frac{A_x}{E} \eta s_a + \frac{B_x}{E} \eta s_b + \frac{C_x}{E} \eta s_c \tag{41}$$

The probable error in position is then

$$\sigma r = \frac{\eta}{E}\sqrt{\left(A_x s_a + B_x s_b + C_x s_c\right)^2 + \left(A_y s_a + B_y s_b + C_y s_c\right)^2 + \left(A_z s_a + B_z s_b + C_z s_c\right)^2} \tag{42}$$

The propagation-error factor may be defined as follows:

$$\begin{aligned}\delta &= \frac{\sigma r}{\eta \times 10^6} \\ &= \frac{10^{-6}}{E}\sqrt{\left(A_x s_a + B_x s_b + C_x s_c\right)^2 + \left(A_y s_a + B_y s_b + C_y s_c\right)^2 + \left(A_z s_a + B_z s_b + C_z s_c\right)^2}\end{aligned} \tag{43}$$

Contour plots of this error factor are shown in figure 10 for the equilateral-triangle configuration. If the error in n were a constant value of 0.0001, and an error factor δ of 7 could be achieved, then σr would equal 700 feet.

The variation of the index of refraction with altitude can also cause propagation in a curved path, which will produce errors. This effect is especially important at low elevation angles; however, it is not investigated herein.

Station-Location Errors

Another aspect of the problem is the accuracy of measuring the location of the receiving stations. Appendix D shows that the ratio of probable error in missile position to the probable error in station location is equal to the error factor ϵ or

$$\frac{\sigma r}{\sigma a} = \epsilon \tag{44}$$

This result may also be seen quite simply from an examination of figure 3. Suppose station A were moved a small distance. It is seen that a change in the vector $\vec{a}$ (vector position of A) is the same as a change in the vector $\vec{r}_a$. Hence, a change in $\vec{a}$ must have the same effect as a change in r_a and, consequently, in s_a.

If the error in station location were 5 feet and if the error factor were as great as 50, the error in transmitter location would be only 250 feet.

VELOCITY DETERMINATIONS

The velocity of the transmitter may be obtained in two different ways. In the most direct method the velocity components are obtained from the x, y, and z positions as a function time. In another method the Doppler frequencies are used in conjunction with the transmitter location. Appendix E shows that the velocity components are

$$v_x = \frac{\lambda}{E}\left[A_x\left(f_o - f_a\right) + B_x\left(f_o - f_b\right) + C_x\left(f_o - f_c\right)\right] \tag{45}$$

$$v_y = \frac{\lambda}{E}\left[A_y\left(f_o - f_a\right) + B_y\left(f_o - f_b\right) + C_y\left(f_o - f_c\right)\right] \tag{46}$$

$$v_z = \frac{\lambda}{E}\left[A_z(f_o - f_a) + B_z(f_o - f_b) + C_z(f_o - f_c)\right] \tag{47}$$

where A_x, B_y, and so forth depend on receiving-station configuration and transmitter location. Actually, the two methods differ only by the errors introduced in the data-reduction process and, hence, preference for one method will be determined by this process.

ERRORS IN VELOCITY

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If it is assumed that the position error is negligible, then the probable error in velocity (see appendix F) is given by

$$\sigma v = \lambda \epsilon \sigma f \tag{48}$$

The instantaneous frequency could not be expected to be as accurate as the integrated frequency. If an accuracy of 0.5 cycle per second can be obtained, then velocity data would be within $(5)(10)(0.5) = 25$ feet per second.

CONCLUSIONS

An analysis has shown that a four-station Doppler system is capable of giving position data on a research vehicle to within 1,000 feet (except at low elevation angles) by using a simple radio beacon transmitter whose frequency need not be known accurately. However, Doppler frequencies must be measured very accurately and variations of the index of refraction should be taken into account for best accuracy.

Errors for the equilateral- and isosceles-triangle configurations may be found from the contour plots given; for other configurations, formulas may be used. Infinite errors may be avoided by locating one station inside the triangle formed by the other three stations.

The equilateral-triangle configuration with one station in the center is the best configuration for general use, although the isosceles-triangle configuration with a station on the midpoint of the base will give better results in a specific direction. Other configurations may be satisfactory but should be checked with the error formulas.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., January 10, 1961.

APPENDIX A

EXISTENCE OF INFINITE ERRORS

If all stations are in the x-y plane, it will be shown that infinite error occurs when $D = 0$. Infinite error occurs whenever E, in equation (38), is zero. The determinant E is given by

$$E = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} \frac{\partial s_a}{\partial x} & \frac{\partial s_a}{\partial y} & \frac{\partial s_a}{\partial z} \\ \frac{\partial s_b}{\partial x} & \frac{\partial s_b}{\partial y} & \frac{\partial s_b}{\partial z} \\ \frac{\partial s_c}{\partial x} & \frac{\partial s_c}{\partial y} & \frac{\partial s_c}{\partial z} \end{vmatrix} \tag{A1}$$

By using equation (28) the following determinant may be obtained:

$$E = \begin{vmatrix} \frac{x}{r_a}\left(1 - \frac{x_a}{x} - \frac{r_a}{r}\right) & \frac{y}{r_a}\left(1 - \frac{y_a}{y} - \frac{r_a}{r}\right) & \frac{z}{r_a}\left(1 - \frac{z_a}{z} - \frac{r_a}{r}\right) \\ \frac{x}{r_b}\left(1 - \frac{x_b}{x} - \frac{r_b}{r}\right) & \frac{y}{r_b}\left(1 - \frac{y_b}{y} - \frac{r_b}{r}\right) & \frac{z}{r_b}\left(1 - \frac{z_b}{z} - \frac{r_b}{r}\right) \\ \frac{x}{r_c}\left(1 - \frac{x_c}{x} - \frac{r_c}{r}\right) & \frac{y}{r_c}\left(1 - \frac{y_c}{y} - \frac{r_c}{r}\right) & \frac{z}{r_c}\left(1 - \frac{z_c}{z} - \frac{r_c}{r}\right) \end{vmatrix} \tag{A2}$$

The next step is to factor x, y, and z out of the columns and r_a, r_b, and r_c out of the rows. For the special case when $z_a = z_b = z_c = 0$ (i.e., all stations located in the x-y plane) E becomes

$$E = \frac{xyz}{r_a r_b r_c} \begin{vmatrix} 1 - \frac{x_a}{x} - \frac{r_a}{r} & 1 - \frac{y_a}{y} - \frac{r_a}{r} & 1 - \frac{r_a}{r} \\ 1 - \frac{x_b}{x} - \frac{r_b}{r} & 1 - \frac{y_b}{y} - \frac{r_b}{r} & 1 - \frac{r_b}{r} \\ 1 - \frac{x_c}{x} - \frac{r_c}{r} & 1 - \frac{y_c}{y} - \frac{r_c}{r} & 1 - \frac{r_c}{r} \end{vmatrix} \tag{A3}$$

Subtraction of the last column from the first two gives

$$E = \frac{xyz}{r_a r_b r_c} \begin{vmatrix} -\frac{x_a}{x} & -\frac{y_a}{y} & 1 - \frac{r_a}{r} \\ -\frac{x_b}{x} & -\frac{y_b}{y} & 1 - \frac{r_b}{r} \\ -\frac{x_c}{x} & -\frac{y_c}{y} & 1 - \frac{r_c}{r} \end{vmatrix} \tag{A4}$$

$$E = -\frac{z}{r r_a r_b r_c} \begin{vmatrix} x_a & y_a & r_a - r \\ x_b & y_b & r_b - r \\ x_c & y_c & r_c - r \end{vmatrix} \tag{A5}$$

$$E = -\frac{zD}{r r_a r_b r_c} \tag{A6}$$

Thus, infinite error occurs when $z = 0$ or $D = 0$.

APPENDIX B

TRIANGLE RULE OF INFINITE ERROR

In order to prove the triangle rule of infinite error, a general triangular configuration of stations A, B, and C lying in a plane (fig. 5) is assumed, with the position of station O, the origin, variable with respect to the triangle. From an investigation of the signs of S_a, S_b, and S_c, it will be shown that if O lies in the unshaded regions of figure 5, the determinant D cannot be zero and infinite error cannot exist. When O lies in the shaded regions it will be shown that there exists a surface of transmitter positions which will cause the determinant D to equal zero and give infinite error. A generalization of figure 5 gives the triangle rule of infinite error which may be stated as follows: Infinite error cannot occur if any one station lies within (or on) the triangle formed by lines joining the other three stations; if no station lies within such a triangle, then infinite errors exist.

Properties of S_a, S_b, and S_c

In order to determine the signs of S_a, S_b, and S_c, consider the identity

$$\begin{vmatrix} x_a & y_a & \vec{a} \\ x_b & y_b & \vec{b} \\ x_c & y_c & \vec{c} \end{vmatrix} \equiv 0 \qquad \text{(B1)}$$

which is readily verified (since stations lie in the x-y plane and $z_a = z_b = z_c = 0$) by subtracting from each element in the last column i times the corresponding element in the first column and j times the corresponding element in the second column, where i and j are the unit vectors in the x and y directions, respectively. Thus,

$$S_a\vec{a} + S_b\vec{b} + S_c\vec{c} \equiv 0 \qquad \text{(B2)}$$

where

$$S_a = x_b y_c - x_c y_b \tag{B3}$$

$$S_b = x_c y_b - x_b y_c \tag{B4}$$

$$S_c = x_a y_b - x_b y_a \tag{B5}$$

If $S_a = 0$, then $\frac{y_b}{x_b} = \frac{y_c}{x_c}$ and the origin 0 lies on BC, or BC extended (fig. 5). When $S_a > 0$, point 0 lies on the same side of BC as point A, whereas if $S_a < 0$, point 0 lies on the opposite side of BC. The signs of S_b and S_c may be determined in the same manner to obtain the remaining information in figure 5. Note that inside the triangle S_a, S_b, and S_c are all positive. On any one side of the triangle one of these quantities is zero.

The identity (B2) may be written in the alternate form

$$-\frac{S_b}{S_a + S_c}\vec{b} \equiv \vec{a} + \frac{S_c}{S_a + S_c}(\vec{c} - \vec{a}) \tag{B6}$$

Figure 4 illustrates this identity and shows that if B lies on AC,

$$-\frac{S_b}{S_a + S_c} = 1 \tag{B7}$$

or

$$S_a + S_b + S_c = 0 \tag{B8}$$

However, if B lies below AC as shown in figures 4 and 5 and if 0 lies in the unshaded region below B, then

$$-\frac{S_b}{S_a + S_c} > 1 \tag{B9}$$

or, since S_a and S_c are both negative,

$$S_a + S_b + S_c > 0 \tag{B10}$$

On the other hand, if 0 lies in the shaded region above AC (fig. 5), then

$$-\frac{S_b}{S_a + S_c} < 1 \tag{B11}$$

and since S_a and S_c are both positive,

$$S_a + S_b + S_c > 0 \tag{B12}$$

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In a similar manner, it may be shown that equation (B12) is true in all the other regions (except on the extended sides of the triangle where equation (B8) holds).

Regions of No Infinite Error

It was shown in appendix A that for infinite error the following condition must exist:

$$D = \begin{vmatrix} x_a & y_a & r_a - r \\ x_b & y_b & r_b - r \\ x_c & y_c & r_c - r \end{vmatrix} = 0 \tag{B13}$$

$$D = S_a(r_a - r) + S_b(r_b - r) + S_c(r_c - r) = 0 \tag{B14}$$

or

$$S_a r_a + S_b r_b + S_c r_c = (S_a + S_b + S_c)r \tag{B15}$$

Identity (B2) may be written

$$S_a\left(\vec{r} - \vec{r}_a\right) + S_b\left(\vec{r} - \vec{r}_b\right) + S_c\left(\vec{r} - \vec{r}_c\right) \equiv 0 \tag{B16}$$

or

$$S_a\vec{r}_a + S_b\vec{r}_b + S_c\vec{r}_c \equiv (S_a + S_b + S_c)\vec{r} \qquad \text{(B17)}$$

Since S_a, S_b, and S_c are all positive (as they are when station O lies inside the triangle) it follows that

$$S_a\left|\vec{r}_a\right| + S_b\left|\vec{r}_b\right| + S_c\left|\vec{r}_c\right| \geqq (S_a + S_b + S_c)\left|\vec{r}\right| \qquad \text{(B18)}$$

Hence, equation (B15) cannot be satisfied (except when $\vec{r}_a$, $\vec{r}_b$, and $\vec{r}_c$ are parallel at $r = \infty$), and infinite error cannot exist when station O lies inside the triangle ABC. (This is also true if station O lies on a side of the triangle.)

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In the unshaded region below station B (fig. 5) equation (B15) may be written (in order to emphasize that $S_a < 0$ and $S_b < 0$)

$$S_b r_b = (S_a + S_b + S_c)r + (-S_a)r_a + (-S_b)r_b \qquad \text{(B19)}$$

and equation (B17) becomes

$$S_b\vec{r}_b = (S_a + S_b + S_c)\vec{r} + (-S_a)\vec{r}_a + (-S_b)\vec{r}_b \qquad \text{(B20)}$$

Hence,

$$S_b\left|\vec{r}_b\right| \leqq (S_a + S_b + S_c)\left|\vec{r}\right| + (-S_a)\left|\vec{r}_a\right| + (-S_b)\left|\vec{r}_b\right| \qquad \text{(B21)}$$

and no infinite error can exist. Note that in this region station B is enclosed by the triangle ACO. In a similar manner it can be shown that infinite error cannot exist in the other unshaded regions.

Regions of Infinite Error

In order to show that infinite error exists when station O lies in the shaded regions of figure 5, consider D as a function of x, y, and z (transmitter position). If $D(x,y,z) > 0$ at some point in space and less than zero at some other point in space, then D must be zero on some surface separating these two points (for no matter what path is taken between the two points, D must go through zero somewhere along this path) and infinite error must exist. The two points which show this are the origin O and one of the corners of the triangle. Consider the shaded region to the left of side AB. Then, at the origin r is zero and (since $S_c < 0$) equation (B14) becomes

$$D(0) = S_a r_a + S_b r_b - (-S_c) r_c \tag{B22}$$

and equation (B17) becomes

$$S_a \vec{r}_a + S_b \vec{r}_b = (-S_c) \vec{r}_c \tag{B23}$$

Thus,

$$S_a |\vec{r}_a| + S_b |\vec{r}_b| \geqq (-S_c) |\vec{r}_c| \tag{B24}$$

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and, therefore,

$$D(0) > 0 \tag{B25}$$

At the point A, r_a is zero and equation (B14) becomes

$$D(A) = -(S_a + S_b + S_c) r - (-S_c) r_c + S_b r_b \tag{B26}$$

Equation (B17) becomes

$$S_b \vec{r}_b = (S_a + S_b + S_c) \vec{r} + (-S_c) \vec{r}_c \tag{B27}$$

Since all coefficients are positive

$$S_b |\vec{r}_b| \leqq (S_a + S_b + S_c) |\vec{r}| + (-S_c) |\vec{r}_c| \tag{B28}$$

and, therefore, $D(A) < 0$. Thus, $D = 0$ somewhere between stations 0 and A and infinite error exists. The other shaded areas may be analyzed in a similar manner.

APPENDIX C

INFINITE ERRORS IN A RECTANGULAR CONFIGURATION

For a rectangular configuration, consider figure 6. Infinite error occurs when

$$D = \begin{vmatrix} x_a & y_a & r_a - r \\ x_b & y_b & r_b - r \\ x_c & y_c & r_c - r \end{vmatrix} = \begin{vmatrix} 0 & m & r_a - r \\ l & m & r_b - r \\ l & 0 & r_c - r \end{vmatrix} = 0 \tag{C1}$$

Thus,

$$r + r_b = r_a + r_c \tag{C2}$$

or

$$r^2 + 2rr_b + r_b{}^2 = r_a{}^2 + 2r_ar_c + r_c{}^2 \tag{C3}$$

But

$$r_a{}^2 = r^2 - 2my + m^2 \tag{C4}$$

$$r_b{}^2 = r^2 - 2lx - 2my + l^2 + m^2 \tag{C5}$$

$$r_c{}^2 = r^2 - 2lx + l^2 \tag{C6}$$

When equations (C4), (C5), and (C6) are substituted into equation (C3) the following equations are obtained:

$$rr_b = r_ar_c \tag{C7}$$

or

$$r^2r_b{}^2 = r_a{}^2r_c{}^2 \tag{C8}$$

Then, if equations (C4), (C5), and (C6) are substituted into equation (C8) there results

$$\ell m(\ell - 2x)(m - 2y) = 0 \tag{C9}$$

Infinite error therefore occurs if

$$x = \frac{\ell}{2} \tag{C10}$$

or

$$y = \frac{m}{2} \tag{C11}$$

Thus, infinite error occurs on the planes which are perpendicular bisectors of the sides of the rectangle.

APPENDIX D

STATION-LOCATION ERRORS

In order to determine the errors in position due to errors in station location, the following range differences, which follow the definitions, may be examined:

$$s_a = \sqrt{(x - x_a)^2 + (y - y_a)^2 + (z - z_a)^2} - \sqrt{x^2 + y^2 + z^2} \tag{D1}$$

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1
2
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$$s_b = \sqrt{(x - x_b)^2 + (y - y_b)^2 + (z - z_b)^2} - \sqrt{x^2 + y^2 + z^2} \tag{D2}$$

$$s_c = \sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2} - \sqrt{x^2 + y^2 + z^2} \tag{D3}$$

These equations may be interpreted as follows: s_a, s_b, and s_c are measured input quantities; x_a, y_a, and so forth are constants depending on station locations; x, y, and z are the unknowns to be found.

If it is assumed that exact data s_a, s_b, and s_c are being received, x, y, and z could be computed exactly if x_a, y_a, z_a; x_b, y_b, z_b; . . . are known exactly. But if there are errors in measuring x_a, y_a, and so forth the problem is then to find the corresponding errors in x, y, and z. The procedure is to find the variations in x, y, and z due to variations in x_a, y_a, and so forth with s_a, s_b, and s_c held constant. Taking the total differential of s_a in equation (D1) and setting it equal to zero gives

$$ds_a = 0$$

$$= \frac{\partial s_a}{\partial x_a} dx_a + \frac{\partial s_a}{\partial y_a} dy_a + \frac{\partial s_a}{\partial z_a} dz_a + \frac{\partial s_a}{\partial x} dx + \frac{\partial s_a}{\partial y} dy + \frac{\partial s_a}{\partial z} dz \tag{D4}$$

But,

$$\frac{\partial s_a}{\partial x_a} = -\frac{x - x_a}{r_a} \tag{D5}$$

$$\frac{\partial s_a}{\partial y_a} = - \frac{y - y_a}{r_a} \tag{D6}$$

$$\frac{\partial s_a}{\partial z_a} = - \frac{z - z_a}{r_a} \tag{D7}$$

Substituting equations (D5), (D6), and (D7) and a_x, a_y, and a_z for $\frac{\partial s_a}{\partial x}$, $\frac{\partial s_a}{\partial y}$, and $\frac{\partial s_a}{\partial z}$ into equation (D4) gives

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$$a_x\,dx + a_y\,dy + a_z\,dz = \frac{x - x_a}{r_a}\,dx_a + \frac{y - y_a}{r_a}\,dy_a + \frac{z - z_a}{r_a}\,dz_a \tag{D8}$$

Similarly,

$$b_x\,dx + b_y\,dy + b_z\,dz = \frac{x - x_b}{r_b}\,dx_b + \frac{y - y_b}{r_b}\,dy_b + \frac{z - z_b}{r_b}\,dz_b \tag{D9}$$

$$c_x\,dx + c_y\,dy + c_z\,dz = \frac{x - x_c}{r_c}\,dx_c + \frac{y - y_c}{r_c}\,dy_c + \frac{z - z_c}{r_c}\,dz_c \tag{D10}$$

These equations may be interpreted as follows: dx, dy, and dz are the errors in x, y, and z, respectively, caused by the errors dx_a, dy_a, and so forth.

Solving equations (D8), (D9), and (D10) for dx yields

$$\begin{aligned} dx = \frac{1}{E}\Bigg[&\left(\frac{x - x_a}{r_a}\,dx_a + \frac{y - y_a}{r_a}\,dy_a + \frac{z - z_a}{r_a}\,dz_a\right)A_x + \left(\frac{x - x_b}{r_b}\,dx_b\right. \\ &\left. + \frac{y - y_b}{r_b}\,dy_b + \frac{z - z_b}{r_b}\,dz_b\right)B_x + \left(\frac{x - x_c}{r_c}\,dx_c + \frac{y - y_c}{r_c}\,dy_c\right. \\ &\left. + \frac{z - z_c}{r_c}\,dz_c\right)C_x\Bigg] \end{aligned} \tag{D11}$$

The probable errors in x_a, y_b, and so forth may be expected to be equal but uncorrelated. Thus, letting σa be the probable error in any one of the quantities yields

$$\sigma x_a = \sigma x_b = \sigma x_c = \sigma y_a = \sigma y_b = \sigma y_c = \sigma z_a = \sigma z_b = \sigma z_c = \sigma a \tag{D12}$$

The probable error in x is the square root of the sum of the squares of each error term. Thus,

$$\sigma x = \frac{\sigma a}{E}\sqrt{\left[\left(\frac{x - x_a}{r_a}\right)^2 + \left(\frac{y - y_a}{r_a}\right)^2 + \left(\frac{z - z_a}{r_a}\right)^2\right]A_x^2 + \left[\left(\frac{x - x_b}{r_b}\right)^2 + \left(\frac{y - y_b}{r_b}\right)^2 + \left(\frac{z - z_b}{r_b}\right)^2\right]B_x^2 + \left[\left(\frac{x - x_c}{r_c}\right)^2 + \left(\frac{y - y_c}{r_c}\right)^2 + \left(\frac{z - z_c}{r_c}\right)^2\right]C_x^2} \tag{D13}$$

Substituting equations (10), (11), and (12) into equation (D13) gives

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$$\sigma x = \frac{\sigma a}{E}\sqrt{A_x^2 + B_x^2 + C_x^2} \tag{D14}$$

In a similar manner,

$$\sigma y = \frac{\sigma a}{E}\sqrt{A_y^2 + B_y^2 + C_y^2} \tag{D15}$$

$$\sigma z = \frac{\sigma a}{E}\sqrt{A_z^2 + B_z^2 + C_z^2} \tag{D16}$$

Since

$$\sigma r = \sqrt{\sigma x^2 + \sigma y^2 + \sigma z^2} \tag{D17}$$

equations (D14), (D15), and (D16) combine to become

$$\frac{\sigma r}{\sigma a} = \frac{1}{E}\sqrt{A_x^2 + B_x^2 + C_x^2 + A_y^2 + B_y^2 + C_y^2 + A_z^2 + B_z^2 + C_z^2} \tag{D18}$$

Since the right side of equation (D18) is identical to the right side of equation (38),

$$\frac{\sigma r}{\sigma a} = \epsilon$$

APPENDIX E

DETERMINATION OF VELOCITY

In order to find expressions for the velocity, consider derivatives with respect to time instead of differentials in equations (25), (26), and (27) as follows:

$$\frac{ds_a}{dt} = a_x \frac{dx}{dt} + a_y \frac{dy}{dt} + a_z \frac{dz}{dt} \tag{E1}$$

$$\frac{ds_b}{dt} = b_x \frac{dx}{dt} + b_y \frac{dy}{dt} + b_z \frac{dz}{dt} \tag{E2}$$

$$\frac{ds_c}{dt} = c_x \frac{dx}{dt} + c_y \frac{dy}{dt} + c_z \frac{dz}{dt} \tag{E3}$$

Solving for the x component of velocity gives

$$\frac{dx}{dt} = \frac{1}{E}\left(A_x \frac{ds_a}{dt} + B_x \frac{ds_b}{dt} + C_x \frac{ds_c}{dt}\right) \tag{E4}$$

However, differentiation of equation (7) gives

$$\frac{ds_a}{dt} = (f_o - f_a)\lambda \tag{E5}$$

Thus,

$$v_x = \frac{dx}{dt} = \frac{\lambda}{E}\left[A_x(f_o - f_a) + B_x(f_o - f_b) + C_x(f_o - f_c)\right] \tag{E6}$$

The other components may be found in a similar manner.

APPENDIX F

ERRORS IN VELOCITY

The error in v_x caused by errors in frequency measurement is obtained from equation (E6) (if it is assumed that position is known accurately) as follows:

$$dv_x = \frac{\lambda}{E}\left[A_x\, d(f_o - f_a) + B_x\, d(f_o - f_b) + C_x\, d(f_o - f_c)\right] \tag{F1}$$

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Now if it is assumed that the probable errors $\sigma(f_o - f_a)$, $\sigma(f_o - f_b)$, and $\sigma(f_o - f_c)$ are all independent and equal to σf, the equation may be written

$$v_x = \frac{\lambda}{E}\,\sigma f\sqrt{{A_x}^2 + {B_x}^2 + {C_x}^2} \tag{F2}$$

Similarly,

$$v_y = \frac{\lambda}{E}\,\sigma f\sqrt{{A_y}^2 + {B_y}^2 + {C_y}^2} \tag{F3}$$

$$v_z = \frac{\lambda}{E}\,\sigma f\sqrt{{A_z}^2 + {B_z}^2 + {C_z}^2} \tag{F4}$$

The probable error in $\vec{v}$ is then

$$\sigma v = \frac{\lambda\sigma f}{E}\sqrt{{A_x}^2 + {B_x}^2 + {C_x}^2 + {A_y}^2 + {B_y}^2 + {C_y}^2 + {A_z}^2 + {B_z}^2 + {C_z}^2} \tag{F5}$$

Thus,

$$\sigma v = \lambda\epsilon\sigma f$$

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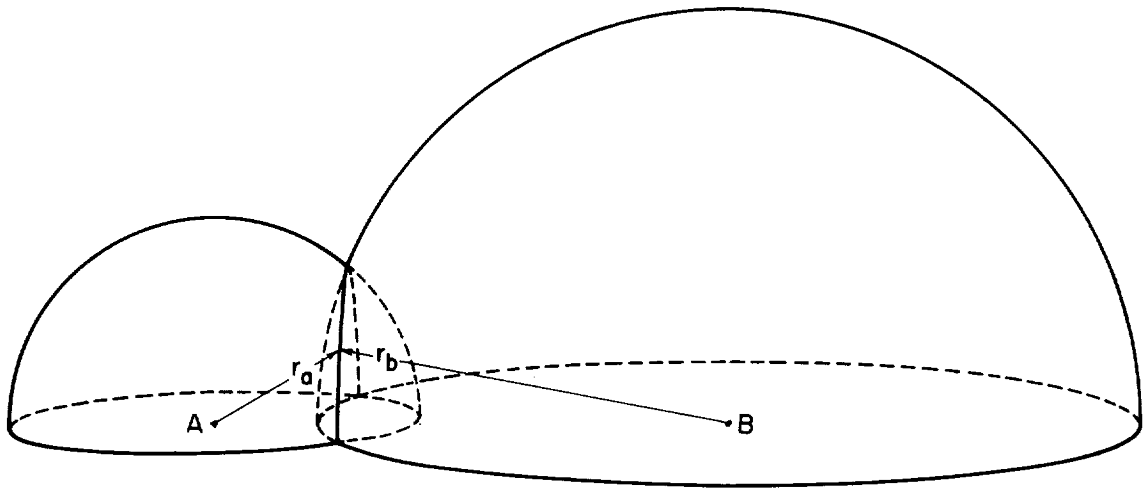

Figure 1.- Two stations of a graphical three-station solution.

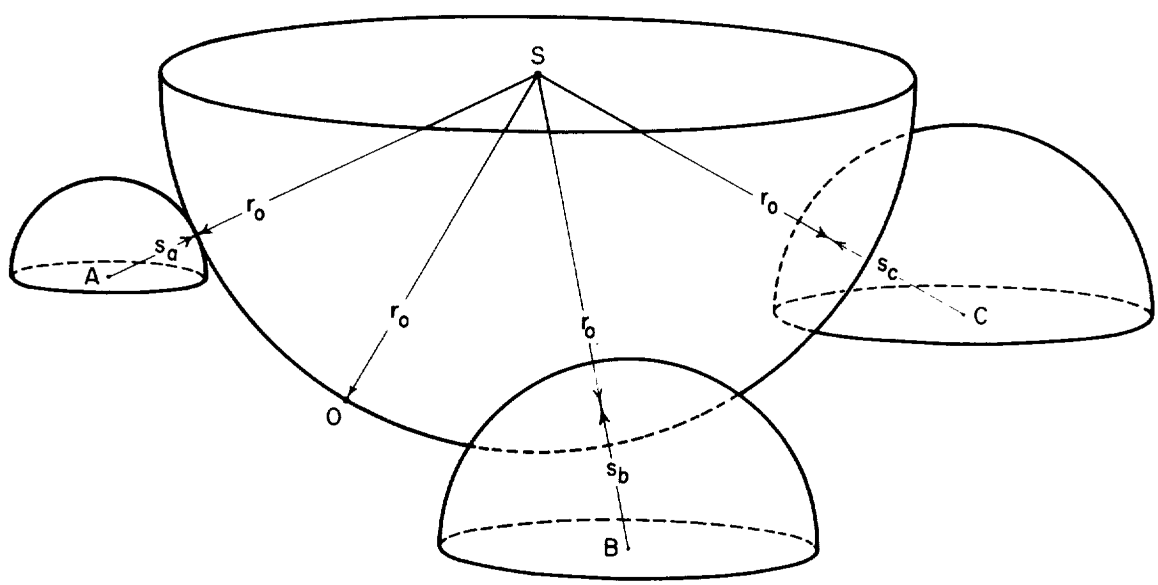

Figure 2.- Graphical four-station solution.

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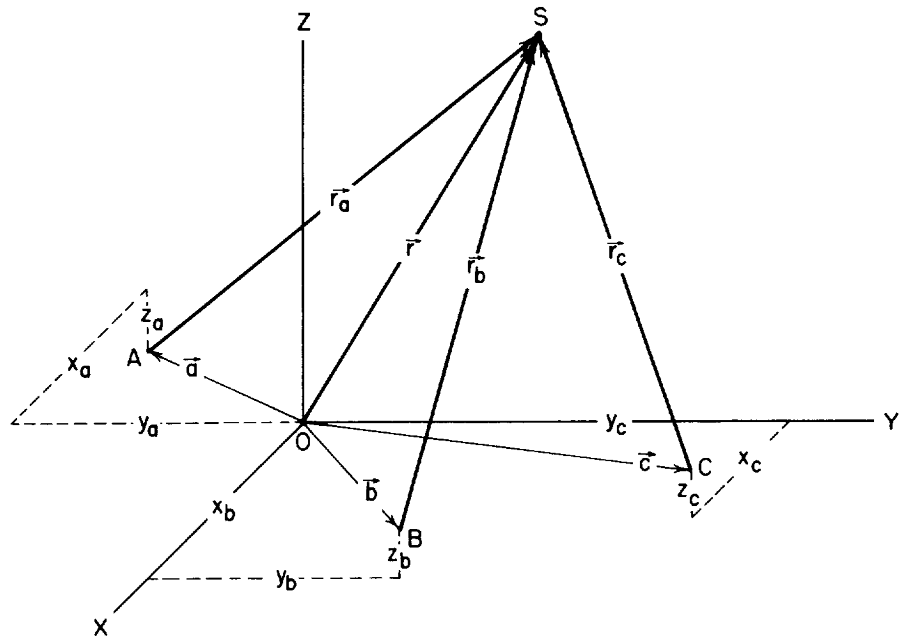

Figure 3.- Geometry of a four-station system.

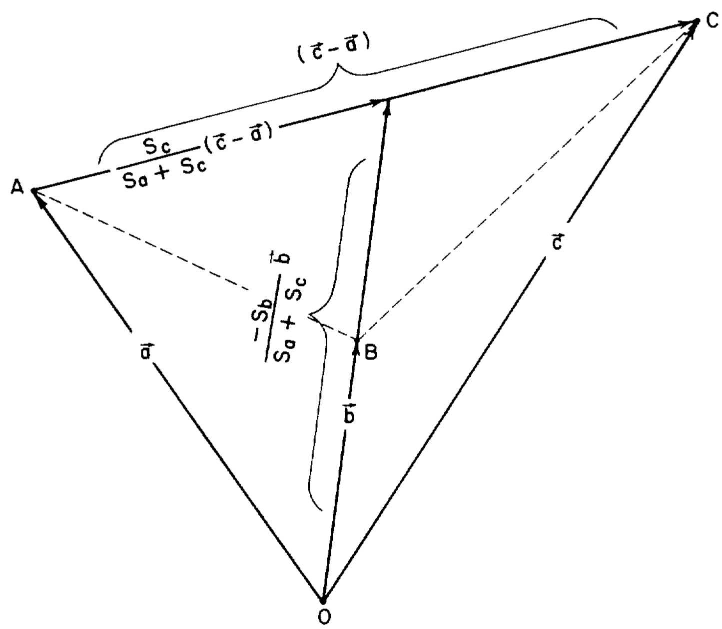

Figure 4.- Vector diagram illustrating identity (B6). If B lies on AC, then $\frac{-S_b}{S_a + S_c} = 1$.

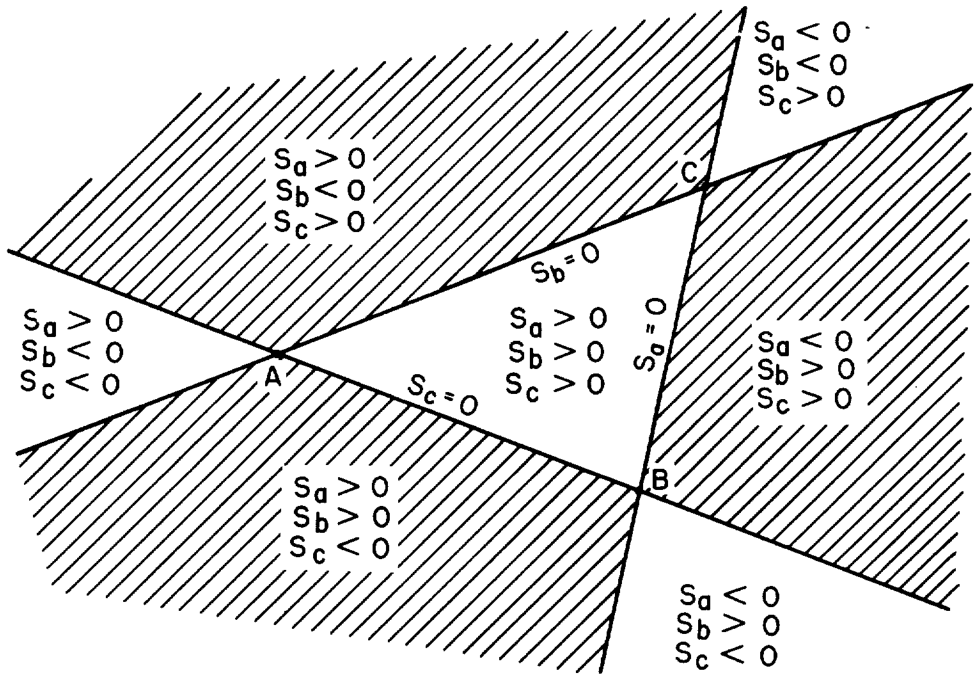

Figure 5.- Illustration of triangle rule of infinite error. When station O lies in the shaded regions, infinite error exists; otherwise no infinite error exists.

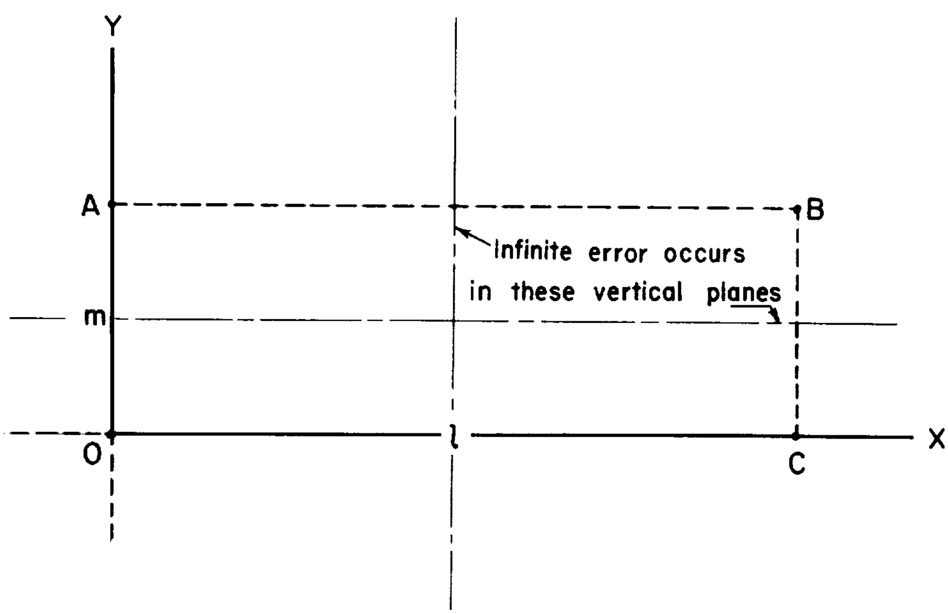

Figure 6.- Rectangular configuration showing planes of infinite error.

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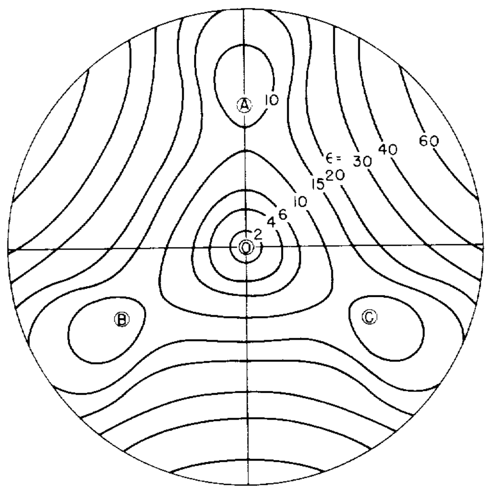
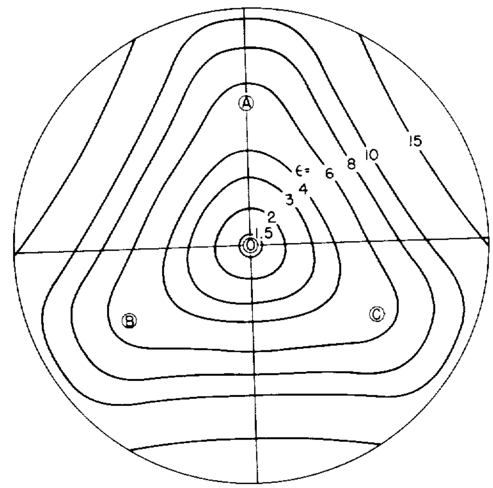
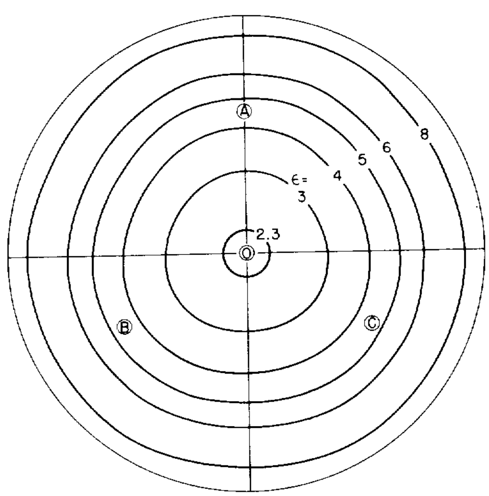


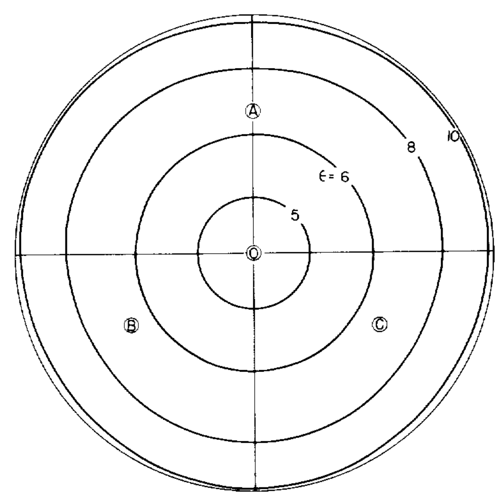
(a) Altitude, 50,000 feet.



(b) Altitude, 150,000 feet.

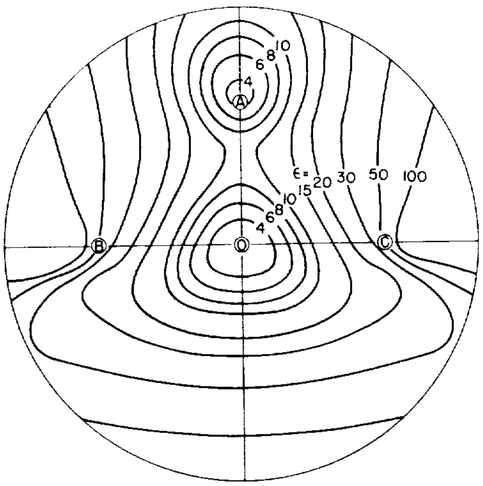


(c) Altitude, 500,000 feet.

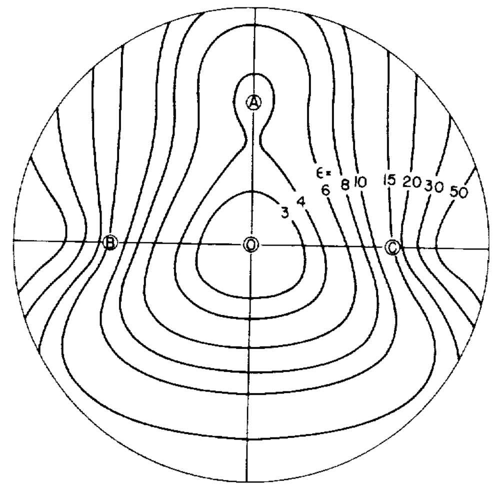


(d) Altitude, 1,000,000 feet.

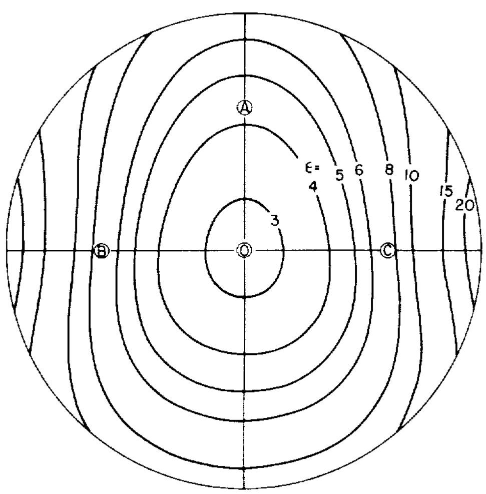
Figure 7.- Error-factor contour plots for equilateral-triangle configuration. Radii of charts, 1×10^6 feet.



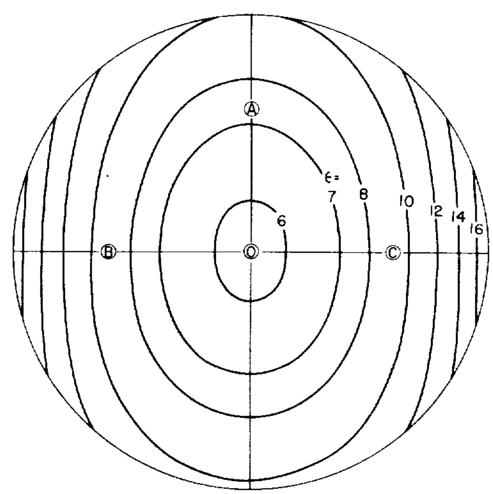
(a) Altitude, 50,000 feet.



(b) Altitude, 150,000 feet.



(c) Altitude, 500,000 feet.



(d) Altitude, 1,000,000 feet.

Figure 8.- Error-factor contour plots for isosceles-triangle configuration. Radii of charts, 1×10^6 feet.

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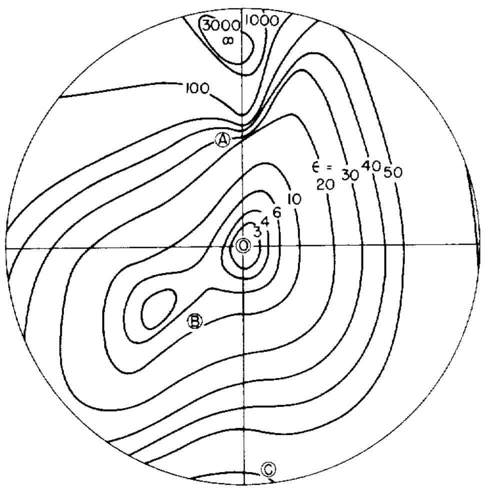
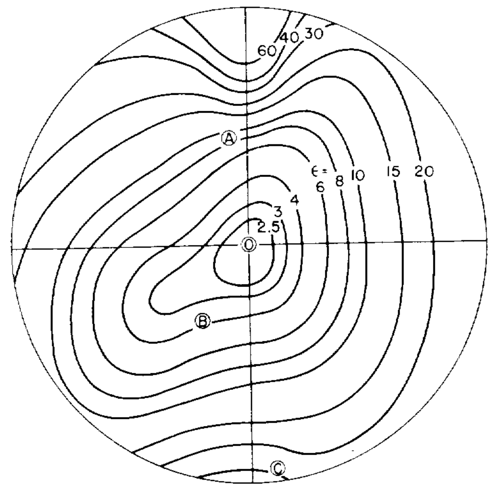
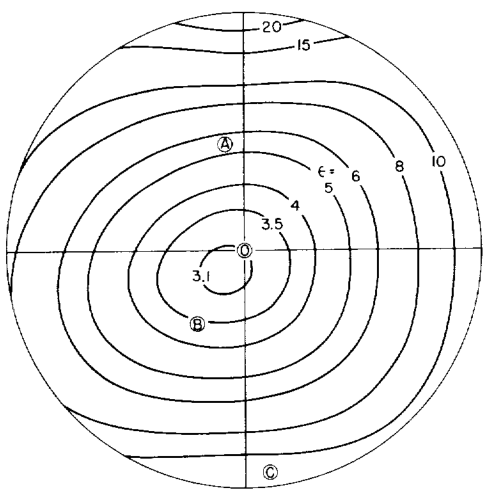


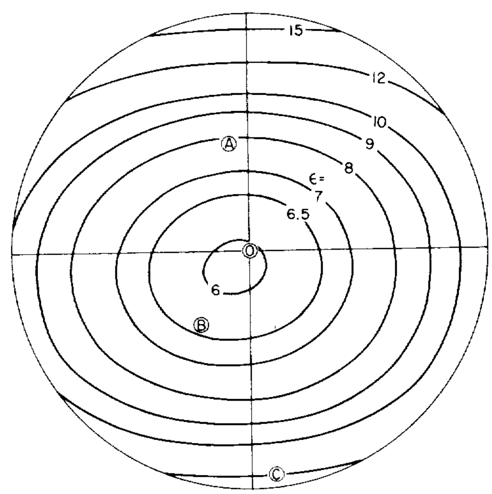
(a) Altitude, 50,000 feet.



(b) Altitude, 150,000 feet.

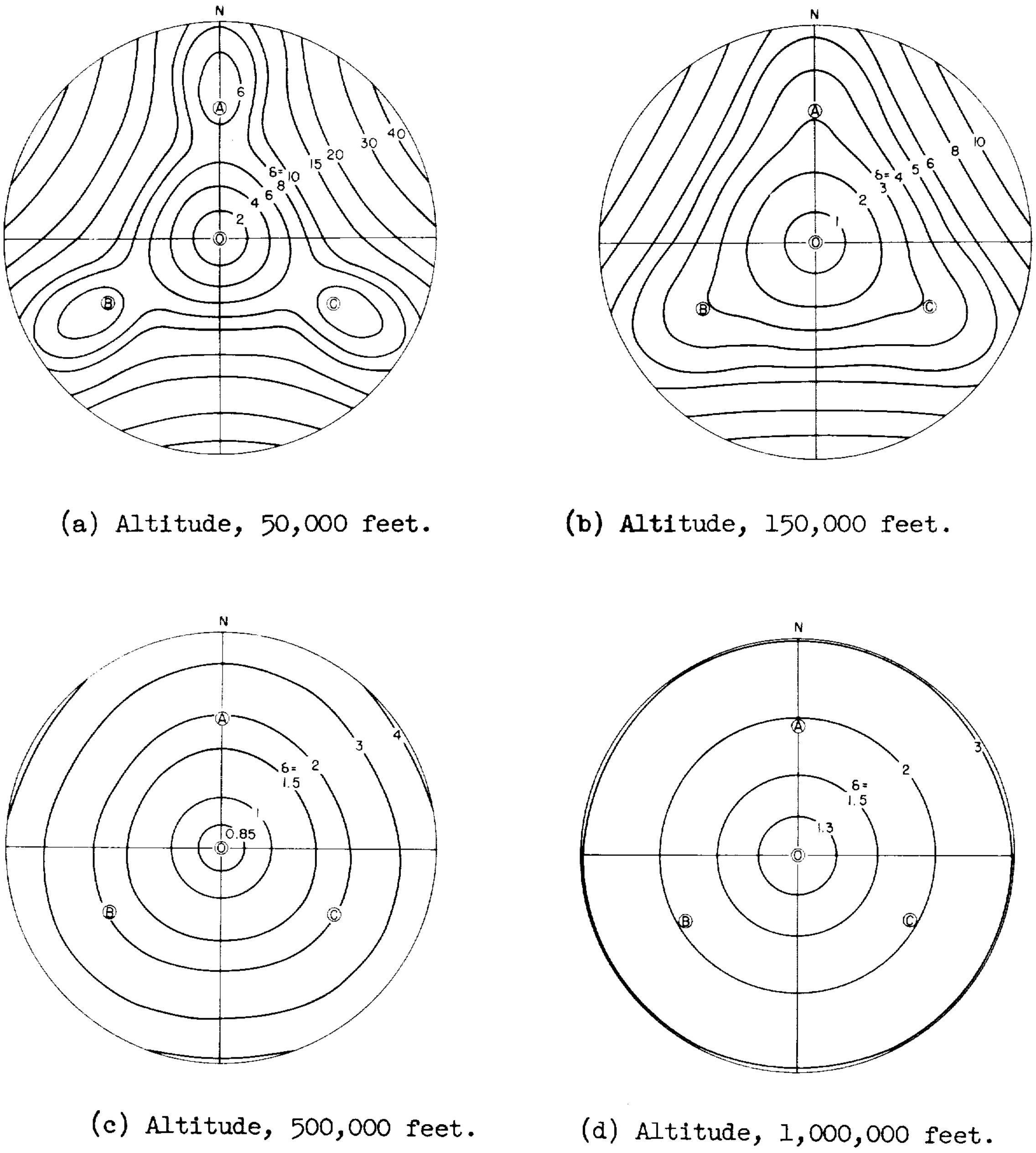


(c) Altitude, 500,000 feet.



(d) Altitude, 1,000,000 feet.

Figure 9.- Error-factor contour plots for particular configuration. Radii of charts, 1×10^6 feet. Coordinates of stations in 1,000 feet (origin at Wallops Island, Va.): station A, Dover, Del. (-80, 454, -5); station B, Langley Field, Va. (-207, -310, -3); station C, Hatteras, N.C. (97, -940, -21).



(a) Altitude, 50,000 feet.

(b) Altitude, 150,000 feet.

(c) Altitude, 500,000 feet.

(d) Altitude, 1,000,000 feet.

Figure 10.- Propagation-error-factor contour plots for equilateral-triangle configuration. Radii of charts, 1×10^6 feet.

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